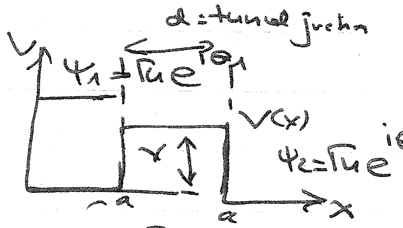
QUANTUM OPTICS : SUPERCONDUCTING (1) CIRCUITS.

Basics of superconducty JJ

d: tunnel junction



Bond

$\theta_1 - \theta_2 = \Delta\theta$: phase accross junction

Macroscopic Quantum Model: $q^* = 2\cdot e$

London Thy:

λ_L = pen. depth.

$$\boxed{\psi_*(\vec r,t) = \sqrt{n}\cdot e^{i\varphi(\vec r,t)}}$$ wave function

$|\psi(r,t)|^2 = n(r,t)$: electron - cooper pair density

θ: gauge invariant phase

Schrödinger equation of electron in mgn. & electric field.

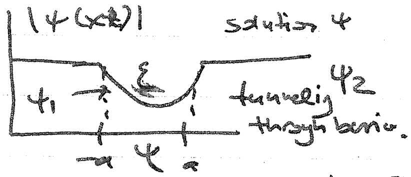
V: potential

$$\boxed{i\hbar\partial_t\psi = \frac{1}{2m^*}(\underbrace{\vec p}_{-i\hbar\vec\nabla} - q^*\vec A)^2\psi + (\underbrace{\Phi\cdot q^*}_{\text{Potential } V_0 \pm})\cdot\psi}$$ LONDON Theory.

$\Phi_0 = \frac{h}{2e}$

Flux quantum

Schrödingers eq $\begin{cases} q^*: & \text{charge of quasi particle} \\ m^*: & \text{mass of quasi-particles}\end{cases}$



we need this w.fct.

Relate this to a current:

$$i\hbar(\partial_t\psi^*)\psi + (i\hbar\partial_t\psi)\psi^* = i\hbar\partial_t|\psi|^2 = i\hbar\underbrace{\partial_t\rho}_{\text{change in density}} \hat= i\hbar\partial_t \underbrace{n(t)}_{\text{electron density}} \hat= -\vec\nabla\cdot\underbrace{\vec J}_{\text{Probability current}}$$

$\Rightarrow$

ie change in $n(t)$ (density) implies change in current.

$$\Rightarrow \boxed{J_\varphi = \mathrm{Re}\{\psi^*(r,t).\ (\frac{\hbar}{im^*}\vec\nabla - \frac{q^*}{m^*}\vec A)\psi(r,t)\}}$$ Probability current of a charged quantum particle

Calculation:

(using: $\vec\nabla\cdot(f\vec C) = f\vec\nabla\cdot\vec C + \vec C\cdot\vec\nabla f$)

$$\text{i.s}\quad \psi^* i\hbar\partial_t\psi = \frac{-\hbar^2}{2m^*}\psi^*\nabla^2\psi = \frac{-\hbar^2}{2m}(\nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*))$$

$\psi(+a) = \sqrt{n}e^{i\varphi}$

$\psi(-a) = \sqrt{n}e^{i\varphi_1}$

The current flowing through the junction is: (choose gauge $\vec A = 0$)

$$\text{Solve } \frac{\hbar}{2m}\nabla^2\psi = (\mathcal{E}_0 - V)\psi \quad |x| < a \qquad \boxed{J_\varphi = \frac{q^*}{m^*}\mathrm{Re}\{\psi^*(\frac{i\hbar}{*})\nabla\psi\} \hat\Rightarrow \frac{q^*}{m^*}\frac{\hbar}{\xi}\mathrm{Im}\{c_1c_2^*\}}$$

$\psi = c_1\cosh\frac{x}{\xi} + c_2\sinh\frac{x}{\xi}$ is solution to the wave fctn

Phase difference accross junction.

CONNECTER.

$(N_1+N_2)2 = 2N$: PAIRS

$N\varphi_1 + N\varphi_2 = \varphi$

$\psi^{(1,c)} = \psi_0^1\psi_0^2$

$$\Rightarrow \boxed{J_{\text{supercurrent}} = J_{\text{critical}}\sin(\theta_1 - \theta_2)}$$ CURRENT PHASE RELATIONSHIP.

DC JOSEPHSON EFF

$$\boxed{J_{\text{critical}} = -\frac{q^*\hbar}{2m^*\xi}\cdot\frac{\sqrt{n_1^* n_2^*}}{\sinh(2a/\xi)}}$$

CRITICAL CURRENT

$\sinh(2a/\xi) \simeq \frac{e^{2a/\xi}}{2}$ ie exponential.

$J_{\text{crit}} \propto e^{-2a/\xi}$ $\Rightarrow$ smaller for larger distances. smaller

Important: gauge invariance of the phase!

$$\boxed{\xi = \frac{\hbar^2}{2m_e(V_0 - \mathcal{E}_0)}}$$ $\mathcal{E}_0$: Energy of $\psi(r)$

$\psi \to \psi e^{i\chi(r,t)\frac{q}{\hbar}}$

Let's assume that we transform to new gauge $\left|\begin{array}{l} \vec A' \equiv \vec A + \vec\nabla\chi \\ U' \equiv U - \partial_t\chi \end{array}\right|$

$\Rightarrow \vec E = -\nabla\Phi - \frac{\partial A'}{\partial t}$ electrical field

Basics of JJ

[ch. 5. Orlando book]

Key point: the current $J_{super} = q^* n^* \left(\frac{\hbar}{m^*} \nabla\theta' - \frac{q^*}{m^*}\vec{A}' \right)$

(which can be measured) must be the same. (Physical reality)

thus: $\vec{\nabla}(\theta - \theta')\frac{\hbar}{m^*} - \frac{q^*}{m}(\underbrace{A' - A}_{-\vec{\nabla}\chi}) = 0$ i.e. $A = -\vec{\nabla}\chi(\vec{r}, t)$

$\Leftarrow$ $\boxed{\theta' = \theta + \frac{q^*}{\hbar}\cdot\chi}$ $q^* = 2e$

Relation ensures that $J = J'$ i.e. supercurrent is GAUGE INVARIANT.

Hence $\boxed{\psi'(r,t) = \psi(r,t)\cdot e^{i\, q^*/\hbar \cdot \chi} = e^{i\frac{2e}{\hbar}\cdot\chi}}$ $\boxed{2\pi/\Phi_0 \equiv \frac{2e}{\hbar}}$

$\rightarrow$ Change of A leads to change of wave function phase. $\Rightarrow \boxed{\Phi_0 = \frac{h}{2e}}$

Current phase relationship modification for an electrical field

using the relation: $\chi(r_1) - \chi(r_2) \hat{=} -\int_{r_1}^{r_2} \vec{\nabla}\chi\, d\vec{l}$, $\vec{\nabla}\chi = -A$

$\Delta\theta' = \Delta\theta - \frac{2\pi}{\Phi_0}\underbrace{\Delta\chi}_{\chi_1 - \chi_2}$ and $\chi_1 - \chi_2 = \int_1^2 \vec{\nabla}\chi\, d\vec{l} \int_1^2 \vec{A}\, d\vec{l}$

GAUGE INVARIANT PHASE DIFFERENCE

since $\vec{\nabla}\chi = \vec{A}$

$\Rightarrow \boxed{\varphi = (\theta_1 - \theta_2) - \frac{2\pi}{\Phi_0}\cdot\left(\int_1^2 \vec{A}\, d\vec{l}\right)}$ $\left[\oint \vec{A}\, d\vec{l} = \Phi \text{ FLUX}\right]$

Phase shift φ due to magnetic flux Φ: $\varphi = 2\pi\frac{\Phi}{\Phi_0}$

~~V_{12} Potential~~

Time derivative d/dt

DC-JOSEPHSON EFFECT

$\Leftrightarrow \boxed{\frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0}\int_1^2 (\partial_t \vec{A})\, d\vec{l} = \frac{2\pi}{\Phi_0}\cdot V_{12}(t)}$

VOLTAGE PHASE RELATIONSHIP

$\Rightarrow \boxed{\frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0}\cdot V_{12}(t)}$

VOLTAGE PHASE RELATIONSHIP.

JJ

$I_j = I_c \sin\varphi$

$\frac{d\varphi}{dt} = \left(\frac{\Phi_0}{2\pi}\right)\cdot V$

ie $V = \frac{d\varphi}{dt}\frac{\Phi}{2\pi}$

when current is changed also φ changes and a voltage appears $(d\varphi/dt)$ $\Rightarrow$ work is DONE

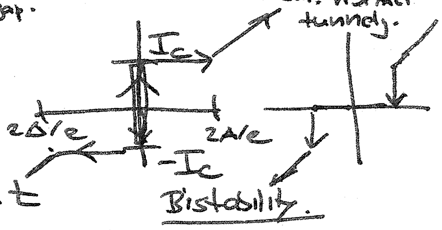
$E_J \hat{=} W_j = \int_0^t i(t)V(t) = \int_0^t I_c\sin\varphi\cdot\left(\frac{\Phi_0}{2\pi}\frac{d(\varphi)}{dt}\right)\cdot dt \underset{\varphi(t)}{=} \int_0^{t(\varphi)} I_c\sin\varphi\, d\varphi \cdot \frac{\Phi_0}{2\pi}$

$\boxed{E_J \hat{=} \left| W_j = \underbrace{\frac{I_c\Phi_0}{2\pi}}_{W_{Josephson}}\cos(\varphi)\right.}$ JOSEPHSON ENERGY $\Downarrow$ needed for Quantization.

ca. 10^{-11} Joule

$V q^* = (2e)V = \Delta$ sup. gap.

$V = \frac{\Delta}{(2e)}$



Bistability.

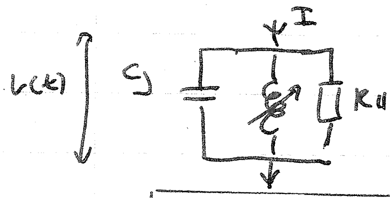
Dynamics at junction $V(t) = V_0 = $ constant. $\varphi(t) = \varphi(0)\cdot + \frac{2\pi}{\Phi_0}\cdot V_0\cdot t$

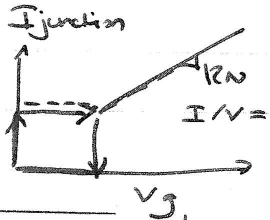
$\Rightarrow \boxed{I_j = I_c\cdot\sin\left(\frac{2\pi}{\Phi_0}V_0 t + \varphi(0)\right)}$ AC JOSEPHSON EFFECT

$f_J = \frac{2\pi}{\Phi_0}V_0 = 4.83\times10^{12}\ \text{Hz/V}$ $\boxed{10\mu V \hat{=} 5\,GHz}$

$= 483\ \frac{THz}{Volt}$ allows determining e/h from freq. voltage relation.

Josphewson junctions





$I_R(t) = \frac{V(t)}{R_N}$

Basis for JJ based logic
→ bistable state. R-SFQ.
(rapid single flux quanta)

$$\boxed{L_j = \frac{L_{j_0}}{\cos\varphi(t)} \wedge L_{j_0} = \frac{\Phi_0}{2\pi}\frac{1}{I_c}}$$

JOSEPHSON INDUCTANCE

Since: $I_j = I_c \cdot \sin\varphi(t)$
$\frac{d\varphi}{dt} = \left(\frac{\Phi_0}{2\pi}\right)^{-1} V(t)$
$\Big\}$ $\Rightarrow V(t) = \frac{\Phi_0}{2\pi}\cdot\frac{d\varphi}{dt} = \frac{\Phi_0}{2\pi}\cdot$ (*)

$\Rightarrow \frac{\partial I}{dt} = I_c[\cos\varphi(t)]\overset{*}{\frac{d\varphi}{dt}} \hat{=} I_c\cdot\cos\varphi\left(\frac{2\pi}{\Phi_0}\right)V(t) \Rightarrow \frac{\partial I}{\partial t}\frac{\Phi_0/2\pi}{\cos\varphi\, I_c} = V$

and $L\frac{\partial I}{dt} \hat{=} +V(t)$ Inductance $\Rightarrow$ $\boxed{L = \frac{\Phi_0}{2\pi\cdot\cos\varphi}\frac{1}{I_c} = \underbrace{\left(\frac{\Phi_0}{2\pi I_c}\right)}_{L_{J_0}}\left(\frac{1}{\cos\varphi}\right)}$

Josephson inductance

$$\boxed{L_{j_0} = \left(\frac{\Phi_0}{2\pi}\frac{1}{I_c}\right)} \qquad \boxed{L_j = \frac{L_{j_0}}{\cos\varphi}}$$

inductance varies with the phase $\varphi(t)$ across junction.

$\left(\frac{\Phi_0}{2\pi} = L_{j_0} I_{crit}\right)$ | flux divided by critical current)

Equation of motion of a JJ

Anharmonic oscillator

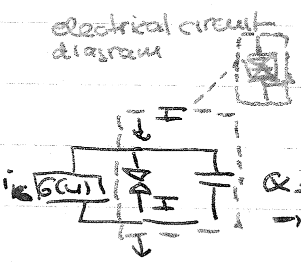
$$* \quad \boxed{I(t) = I_c\sin\varphi(t) + L_{j_0}\cdot C_j\frac{\partial^2\varphi}{\partial t^2}I_c + \frac{L_{j_0}}{R_{II}}I_c\frac{\partial\varphi}{\partial t}}$$

"RSFQ: Rapid Single Flux Quantum"

Foundry:
IPht-Jena
CrypConnComplexity
EUCAS
Conference.

since Generalized junction model

electrical circuit diagram



$i = \underbrace{I_c\sin\varphi}_{JJ} + \underbrace{\frac{dV}{dt}C_j}_{i_c} + \underbrace{V\cdot G(v)}_{\text{conducting}}$

(Kirchhoffs law)

e.g. $G(v) = \frac{1}{R_j}$

$Q = C\cdot V$
$\rightarrow \frac{dQ}{dt} = i_c = C\cdot\frac{dV}{dt}$
$\Leftrightarrow \frac{dQ}{dt} = C\cdot\frac{\Phi_0}{2\pi}\frac{d^2\varphi}{dt^2} = L_j C I_c\cdot\frac{d\varphi}{dt^2}$

$$G(v) = \begin{cases} 0 & |V| < \frac{2\Delta_0}{e} \\ \frac{1}{R_j} & |V| > \frac{2\Delta_0}{e} \end{cases}$$

Junction resistance.

$$\Leftarrow \boxed{I(t) = I_c\cdot\sin\varphi. + C_j\cdot\left(\frac{d^2\varphi}{dt^2}\cdot\underbrace{\frac{\Phi_0}{2\pi}}_{\hat{=} L_{j_0} I_c}\right) + \cdot\frac{d\varphi}{dt}\underbrace{\left(\frac{\Phi_0}{2\pi}\right)}_{L_j I_c}\cdot\frac{1}{R}}$$

q.e.d

Junction Turn-on time:

$I_g \simeq \sin\varphi I_c + \underbrace{(L_{j_0}\cdot C_j\, I_c)}_{T^2_{turn-on}}\frac{d^2\varphi}{dt^2}$

$\Delta W \simeq 10^{-18} J$ Switching
$I_g^2 R_{sub-gate} \simeq 100 nW$

characteristic turn-on time. $\Rightarrow \boxed{T_{turn-on} = \sqrt{\frac{C_j\Phi_0}{2\pi I_c}}}$

$\sqrt{L_s C} = T_{turn-on}$

$L_s = \frac{\Phi}{2\pi I_c}$

$\Bigg\{$
Nb: 4K
$\Delta/2e \simeq 1mV$
$C \simeq 1pF$
$I_g = 1mA$
$\rightarrow T_{turn-on} \simeq 2ps$

(VI) CPB

-13-(5)

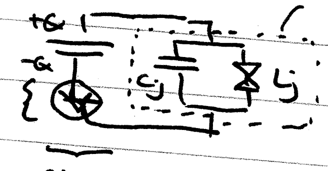
Quantum Optics : THE TRANSMON
COOPER PAIR BOX

1

Solution at perturbation theory:

$\Phi = -\int_{-\infty}^{t} V(t')dt'$ Flux

and $-\frac{d\Phi}{dt} = V$



Energy in junction $L_J = \frac{L_{J0}}{\cos\varphi}$ $E_J = L_{J0} I_c^2 \cos\varphi$ φ: superconducting phase difference

$$\boxed{\hat{H} = \underbrace{\frac{\hat{Q}^2}{2C_{tot}}}_{\text{charging energy}} + \underbrace{L_{J0} \cdot I_c^2}_{E_J} \cdot \cos(\varphi) \triangleq \frac{Q^2}{C_{tot}} + \underbrace{\left(I_c \cdot \frac{\Phi_0}{2\pi}\right)}_{E_J} \cos(\varphi)}$$

$U_L = I_c \cdot \frac{\Phi_0}{2\pi} \cdot \cos(\varphi(t))$ and $\boxed{L_J \triangleq \frac{\Phi_0}{2\pi} \frac{1}{I_{crit}}}$ h: Planck constant $\Phi_0 = \frac{h}{2e}$

Important relation: phase accross junction and Flux: Josephson relation

$$\boxed{\frac{d\hat{\varphi}}{dt} = \frac{2\pi}{\Phi_0} \cdot \hat{V}(t) \quad \text{i.e} \quad \frac{d\hat{\varphi}}{dt} = \frac{2e}{\underbrace{(h/2\pi)}_{\hbar}} \cdot \hat{V}(t)}$$

From this it follows that

$$\boxed{\hat{\varphi}(t) = \frac{2\pi}{\Phi_0} \underbrace{\int_{-\infty}^{t} \hat{V}(t')dt'}_{\hat{\Phi}(t)} = 2\pi \frac{\hat{\Phi}(t)}{\Phi_0}}$$ FLUX-PHASE RELATIONSHIP.

Hence lets re-express the JJ in CHARGE and FLUX:

$$\hat{H} = \frac{\hat{Q}^2}{2C_{tot}} + \left(I_c \frac{\Phi_0}{2\pi}\right) \cdot \cos\left(\frac{2\pi}{\Phi_0} \cdot \hat{\Phi}\right)$$

→ lets now assume that the wavefunction $\psi(\Phi)$ is well localized DESPITE two pi periodicy $|\Phi\rangle \triangleq |\Phi + 2\pi\rangle$. Expand around $\Phi = 0$:

$$\hat{H} = \frac{\hat{Q}^2}{2C_{tot}} + \underbrace{I_c\left(\frac{\Phi_0}{2\pi}\right) \cdot \left(\frac{2\pi}{\Phi_0}\hat{\Phi}\right)^2 \frac{1}{2}}_{\text{Taylor}} + \underbrace{I_c \frac{\Phi_0}{2\pi}}_{\text{constant (drop for Hamiltonian)}}$$

$$= \frac{\hat{Q}^2}{2C_{tot}} + \hat{\Phi}^2 \cdot \left(\frac{I_c 2\pi}{\Phi_0}\right) \cdot \underbrace{\frac{1}{2}}_{\text{Taylor}}$$

$$\boxed{\hat{H} \triangleq \frac{\hat{Q}^2}{2C_{tot}} + \frac{1}{2L_{J0}}\hat{\Phi}^2}$$ Hence $\Omega_J = \sqrt{\frac{1}{C_{tot} L_{J0}}}$ Josephson Plasma frequency.

$\triangleq (a^\dagger a + \frac{1}{2})\hbar\Omega_J$

COOPER PAIR BOX

Josephson Plasma frq $\Omega_j = \sqrt{\frac{1}{L_{jo} C_{tot}}}$. NOTE:

$E_c = \frac{e^2}{2C_{tot}}$: charging energy (also $\frac{(2e)^2}{C_{tot}}$ some authors Devoret!)

$E_j = (I_c \Phi_0) = (I_c L_{jo})^2$ JOSEPHSON ENERGY

Thus: $$\boxed{\hat{H} = \frac{\hat{Q}^2}{C_{tot}} + \frac{\hat{\Phi}^2}{2L_{jo}} \hat{=} 4E_c \underbrace{\left(\frac{\hat{Q}^2}{2e}\right)^2}_{\hat{=}\hat{N}} + \left(\frac{\hat{\Phi}^2}{2}\right)\cdot(E_j)}$$

pair charges COOPER PAIRS

$\hat{N} \equiv \frac{\hat{Q}}{2e}$ CP Number operator.

Hence we have:

$$\Omega_j = \sqrt{\frac{1}{L_{jo} C_{tot}}} \hat{=} \frac{1}{\hbar}\sqrt{8 E_j E_c}$$

since:
$$L_{jo} = \left(\frac{\hbar}{2e}\right)^2 \frac{1}{E_j}$$
$$C_{tot} = \left(\frac{e^2}{2}\right)\frac{1}{E_c}$$

Anharmonicity of the JUNCTION

$\Phi_0 = \frac{h}{2e}$

$$\hat{H} = \frac{\hat{Q}^2}{C_{tot}} + \frac{\hat{\Phi}^2}{2L_{jo}}$$

rewrite in terms of operators $\hat{a}$ and a^+; (recall: LC ho prost)

$$\hat{Q}_{ZPF} = Q_{ZPF}\left(\frac{\hat{a}-\hat{a}^+}{i\sqrt{2}}\right) \qquad Q_{ZPF} = \sqrt{\frac{\hbar}{Z}} \qquad Z = \sqrt{\frac{L}{C}} \hat{=} \frac{1}{\hbar}\sqrt{\frac{8E_c}{8E_j}}\left(\frac{\Phi_0}{2\pi}\right)^2$$
$$\hat{\Phi} = \Phi_{ZPF}\left(\frac{a+a^+}{\sqrt{2}}\right) \qquad \Phi_{ZPF} = \sqrt{\hbar Z}$$

(check next page)

Hence:
$$\hat{\Phi} = \sqrt{\left(\frac{8E_c}{E_j}\right)^{1/2}} \frac{(a+a^+)}{\sqrt{2}} \underbrace{\left(\frac{h}{2e}\right)}_{\frac{\Phi_0}{2\pi}} = \left(\frac{8E_c}{E_j}\right)^{1/4} \frac{(a+a^+)}{\sqrt{2}} \cdot \frac{\Phi_0}{2\pi}$$ Flux operator

hence equivalently:

$$\hat{Q} = 2e\hat{N} = \left(\frac{E_j}{8E_c}\right)^{1/4}\left(\frac{a-a^+}{\sqrt{2}i}\right)\cdot 2e \text{ or: } \hat{N} = \left(\frac{\hat{a}+\hat{a}^+}{i\sqrt{2}}\right)\left(\frac{E_j}{8E_c}\right)^{1/4}$$ NUMBER

Hence: $[\hat{Q}, \hat{\Phi}] = i\hbar$ and $[\hat{N}, \hat{\varphi}] = 1$

Phase operator: $\hat{\varphi} = \varphi_{ZPF}(\hat{a} + \hat{a}^+)$

$$\hat{\varphi} = \frac{2\pi}{\Phi_0}\cdot\hat{\varphi} = \underbrace{\left(\frac{8E_c}{E_j}\right)^{1/4}}_{\varphi_{ZPF}}\frac{(a+a^+)}{\sqrt{2}}$$

ZERO POINT FLUCT. PHASE

PHASE OPERATOR (will discuss later as this is already incorrect!) $|\varphi\rangle = |\varphi + 2\pi\rangle$.

CPB COOPER PAIR BOX

JUNCTION ANHARMONICITY

KOCH ET AL PRA. 2007

with the phase 2 operators:

$$\boxed{\hat{H} = \hbar\Omega_j (\hat{a}^\dagger a + \tfrac{1}{2}) + \underbrace{\frac{\hat{\varphi}^4}{24} E_j}_{\hat{V}\ \text{Perturbation}}}$$

$\Omega_j \equiv \frac{1}{\hbar}\sqrt{8E_jE_c}$ Josephson plasma freq.

Taylor $(1+\hat{\varphi}^2/2+\hat{\varphi}^4/4!)$

Compute effect on the levels $|u\rangle$ 1/12

Thus $\boxed{\hat{V} = -\left(\frac{8E_c}{E_j}\right)\frac{(a+a^\dagger)^4}{(\sqrt{2})^4}\frac{E_j}{24} = -\frac{8}{24\cdot 4}(a+a^\dagger)^4_{E} \simeq \frac{E_c}{12}\left(6\left[a^\dagger a\right]^2 + \hat{a}\hat{a}^\dagger + \tfrac{1}{2}\right)}$

Therefore we compute the Fock state correction

$\Delta E_u \simeq E_c\langle u|\hat{V}|u\rangle$ noting that $\langle u|(a^\dagger)^4|u\rangle = 0$ as $\langle u|\hat{a}^\dagger\hat{a}^3|u\rangle = 0$

Thus:

$\boxed{\Delta E_u \simeq E_c\langle u|(\hat{a}+\hat{a}^\dagger)^4|u\rangle} = \boxed{E_c(6u^2+6u+3)}$ *

normal order $:(a+a^\dagger)^4:$

→ Koch Formula C2 Appendix

NOTE:
only terms with even $\hat{a},\hat{a}^\dagger$ operator contribute

$\langle u|\hat{a}^{\dagger 2}\hat{a}^2|u\rangle = \langle u-1|\hat{a}^\dagger a|u-1\rangle(u) = (u)(u-1)$

* Proof: $\langle u|(\hat{a}+\hat{a}^\dagger)^4|u\rangle = \langle u|\hat{a}^4 + 4\hat{a}^3\hat{a}^\dagger + 3\hat{a}^2\hat{a}^{\dagger 2} + 3\hat{a}^{\dagger 2}\hat{a}^2 + 4\hat{a}^{\dagger 3}\hat{a} + \hat{a}^{\dagger 4}|u\rangle$ → next page.

Therefore the Hamiltonian Eigenspectra becomes:

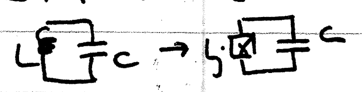
(i) Global shift ("u") term $-\frac{E_c}{12}\cdot(6u)$

$E_u = \hbar\tilde{\Omega}_j\cdot(a^\dagger a + \tfrac{1}{2})$

$= \hbar\left(\sqrt{8E_cE_j} - E_c\right)$

SOFTENING OF POTENTIAL

(ii) ANHARMONICITY (ALSO called SELF-KERR)



$$\boxed{\hat{V} \simeq -\frac{E_c}{2}\cdot(6\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + 2\hat{a}^\dagger a)}$$

NO PROOF.

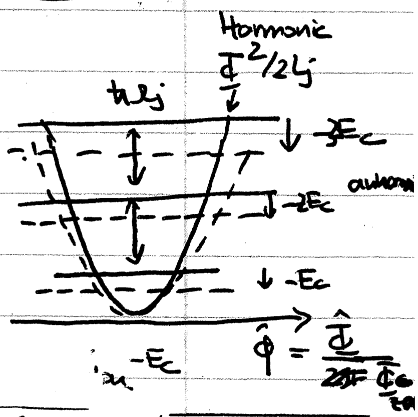
Thus: The Anharmonicity for the levels:

$E_{10} = \langle 1|\hat{H}|1\rangle - \langle 0|\hat{H}|0\rangle \qquad \hat{H} = \hat{H}_0 + \hat{V}$

$E_{21} = \langle 2|\hat{H}|2\rangle - \langle 1|\hat{H}|1\rangle$

becomes

$$\boxed{E_{21} - E_{10} = -E_c}$$



renormalization

$$\boxed{E_u = \underbrace{\left(-E_j + \sqrt{8E_cE_j}\right)(u+\tfrac{1}{2})}_{H.O} - \underbrace{\frac{E_c}{12}(6u^2 + \ldots + 3)}_{\text{nonlinearity}}}$$

$$\boxed{|E_{u+1} - E_u| = \hbar\Omega_j - uE_c}$$

ANHARMONICITY

TRANSMON
COOPER PAIR BOX

Eigenfunctions of the cooper pair box:

$\hat{\varphi}|\varphi\rangle = \varphi|\varphi\rangle$ and $|\varphi + 2\pi\rangle \equiv |\varphi\rangle$ Eigenstates of phase operator

$\hat{N}|N\rangle = N \cdot |N\rangle$: CHARGE STATES / BASIS

Key relationship: $\boxed{|\varphi\rangle = \sum_{N=-\infty}^{\infty} e^{iN\varphi}|N\rangle}$ BLOCH STATES.

From this expression several relationship can be deduced

$e^{\pm i\hat{\varphi}}|N\rangle = |N \pm 1\rangle$

$e^{i\varphi_0\hat{N}}|\varphi\rangle = |\varphi - \varphi_0\rangle$

noting that : (inverse F.T)

$\oint_0^{2\pi} |\varphi\rangle e^{i\tilde{N}\varphi} = 2\pi\, |N\rangle \delta_{N\tilde{N}}$

$\Rightarrow \boxed{|N\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{iN\varphi}|\varphi\rangle}$

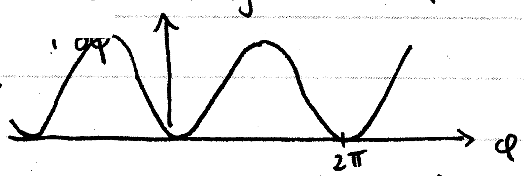
Thus in phase space: (φ)

$\hat{N} = \frac{1}{i}\frac{\partial}{\partial\varphi}$ (differential operator)

or in charge space: (N)

$\hat{\varphi} = i\frac{\partial}{\partial N}$

V/E_J Periodic potential



Therefore: $\hat{H} = 4E_C\hat{N}^2 - E_J\cos\varphi$

E_J/E_C increases localizes φ
$\rightarrow$ increase n distribution

leads to. (using also $(\hat{N} - N_G)^2$: N_G: gate charge) $\psi_\varphi(\varphi) = \langle\varphi|\psi\rangle$ Basis.

$$\boxed{\hat{H}\psi = \left[4E_C \cdot \left(\frac{1}{i}\frac{\partial}{\partial\varphi} - N_G\right)^2 - E_J\cos\varphi\right]\psi_\varphi(\varphi) = E_n\psi_\varphi(\varphi)}$$

[Mathieu Equation]

i.e $\boxed{-4E_C\frac{\partial^2}{\partial\varphi^2}\psi_n(\varphi) - E_J\cos\varphi\,\psi_n(\varphi) = E_n\psi_n(\varphi)}$ MATHIEU EQUATION

One can also compute the CHARGE representation ie not

$\psi_n(\varphi) = \langle\varphi|\psi\rangle$ but $\psi_n(N) = \langle N|\psi\rangle$

in this case solution of:

$$\hat{H}\langle N|\psi\rangle = \hat{H}\psi(N) = (N - N_G)^2 4E_C\psi_N(N) - E_J\cos\left(i\frac{\partial}{\partial N}\right)\psi_N(N) = E_n\psi(N)$$

Proof:

Note: $e^{\pm i\hat{\varphi}}|N\rangle = e^{i\hat{\varphi}}\int_0^{2\pi} e^{iN\varphi} d\varphi|\varphi\rangle = \int_0^{2\pi} e^{(N\pm1)\varphi}|\varphi\rangle d\varphi \,\hat{=}\, |N\pm1\rangle$

likewise

$\hat{N}|\varphi\rangle = \hat{N}\sum_{N=-\infty}^{+\infty} e^{iN\varphi}|\varphi\rangle d\varphi = \sum_N e^{iN\varphi} \cdot N|N\rangle = \frac{1}{i}\frac{d}{d\varphi}\sum e^{iN\varphi}|N\rangle$

APPENDIX COOPER PAIR BOX

Perturbation Theory

$$-\frac{2E_c}{24}(\hat{a}+\hat{a}^+)^4 = -\frac{2E_c}{24}\left(\hat{a}^4 + 4\hat{a}^3\hat{a}^+ + 6\hat{a}^2\hat{a}^{+2} + 3\hat{a}^{+2}\hat{a}^2 + \underbrace{4\hat{a}\hat{a}^{+3}}_{\text{not photon number preserving}} + \hat{a}^4\right)$$

Keeping only terms that conserve photon number

$$= -\frac{2E_c}{24}\left(\underbrace{3\hat{a}^2\hat{a}^{+2}}_{\text{ordering}} + 3\hat{a}^{+2}\hat{a}^2\right)$$

$[\hat{a},\hat{a}^+]=1$

i.e. $\hat{a}\hat{a}^+ = 1+\hat{a}^+\hat{a}$

$$= -\frac{2E_c}{24}\left(3a^{+2}\hat{a}^2 + 3\hat{a}[\hat{a}^+\hat{a}+1]a^+\right)$$

$$= -\frac{2E_c}{24}\left(3\hat{a}^{+2}\hat{a}^2 + 3[\hat{a}\hat{a}^+] + 3\underline{\hat{a}\hat{a}^+}\,\underline{\hat{a}\hat{a}^+}\right)$$

$$= -\frac{2E_c}{24}\left(3\hat{a}^{+2}\hat{a}^2 + (3a^+a+3) + 3[1+\hat{a}^+\hat{a}][1+\hat{a}^+\hat{a}]\right)$$

$$= -\frac{2E_c}{24}\left(3\hat{a}^{+2}\hat{a}^2 + 3a^+a + 6 + 6a^+a + 3\hat{a}^+a\,\hat{a}^+a\right)$$

$$3\hat{a}^+[\hat{a}^+a-1]a$$

$$= -\frac{2E_c}{24}\left[3\hat{a}^{+2}\hat{a}^2 + 6a^+a + 6 + 3\hat{a}^+\hat{a}^+aa\right]$$

$$= -\frac{2E_c}{24}\left(6(\hat{a}^+)^2\hat{a}^2 + 6a^+a + 6\right)$$ NORMAL ORDERING

Easier route to compute corrections

Corrections to E_n:

$$E_n = \langle n|\hat{V}|n\rangle = -\frac{2E_c}{24}\left\{3\langle n|\hat{a}^2(\hat{a}^+)^2|n\rangle + 3\langle n|\hat{a}^{+2}\hat{a}^2|n\rangle\right\}$$

$$= -\frac{2E_c}{24}\left\{3.(\sqrt{n+1})^2(\sqrt{n+2})^2 + 3(\sqrt{n})^2(\sqrt{n-1})^2\right\}$$

$$= -\frac{2E_c}{24}\cdot 3\cdot\left\{(n+1)(n+2) + n(n-1)\right\}$$

$$= -\frac{2E_c}{24}\cdot 3\,(2n^4 + 3n - n + 1)$$

$$= -\frac{2E_c}{24}\,3\,\{2n^4 + 2n + 1)$$

$$= -\frac{2E_c}{24}\{6n^4 + 6n + 3)$$

$$= -\frac{E_c}{12}\{\underbrace{6n^4 + 6n}_{\text{correction of H.O frequency.}} + 3)$$ ($\rightarrow$ Koch PRA C3 equation)

Side remark on definitions: $\Phi \equiv h/2e$ FLUX QUANTUM

$$\hat{\Phi} = \Phi_{ZPF}\frac{(\hat{a}+\hat{a}^+)}{\sqrt{2}} \qquad \Phi_{ZPF} = \sqrt{Z\hbar} = \left[\sqrt{\frac{L}{C}}\right]^{1/2}\sqrt{\hbar} = \sqrt{\hbar}\left(\frac{L}{C}\right)^{1/4} = \left(\frac{\hbar^2 L}{C}\right)^{1/4}$$

and $$\hbar^2\frac{L_J}{C_{tot}} = \hbar^2\left(\frac{\hbar}{2e}\right)^2\frac{1}{E_J}\frac{2E_c}{e^2} = \frac{E_c}{E_J\cdot 2}\cdot\frac{\hbar^4}{e^4} = \frac{E_c 2^3}{E_J}\frac{\hbar^4}{(2e)^4} = \left(\frac{8E_c}{E_J}\right)\left(\frac{\Phi_0}{2\pi}\right)^4$$

$$\Rightarrow \boxed{\hat{\Phi} = \left(\frac{8E_c}{E_J}\right)^{1/4}\frac{\Phi_0}{2\pi}\frac{(\hat{a}+\hat{a}^+)}{\sqrt{2}} \;\hat{=}\; \left(\frac{2E_c}{E_J}\right)^{1/4}\frac{\Phi_0}{2}(\hat{a}+\hat{a}^+)}$$

TRANSMON

COOPER PAIR BOX. $\hat{\varphi}, \hat{N}$ ALGEBRA

Notations:

$$\hat{N}|\varphi\rangle = \sum_N \hat{N}|N\rangle e^{iN\varphi} = \sum_N N e^{iN\varphi}|N\rangle = \frac{1}{i}\frac{\partial}{\partial\varphi} \underbrace{\sum_N e^{iN\varphi}|N\rangle}_{|\varphi\rangle}$$

Thus: $\boxed{\hat{N}|\varphi\rangle = \frac{1}{i\partial\varphi}|\varphi\rangle}$

or. $\boxed{\langle\varphi|\hat{N}|\varphi\rangle = \langle\varphi|\frac{1}{i\partial\varphi}|\varphi\rangle} \hat{=} \frac{i}{\partial\varphi}\langle\varphi|\varphi\rangle$

2nd relation.

$$\hat{\varphi}|N\rangle = \hat{\varphi}\int_0^{2\pi} d\varphi\, e^{iN\varphi}|\varphi\rangle$$

$$= \int_0^{2\pi} d\varphi\, e^{iN\varphi}\, \hat{\varphi}|\varphi\rangle = \int_0^{2\pi} d\varphi\, \varphi\, e^{iN\varphi}|\varphi\rangle$$

$$= \frac{1}{i}\frac{1}{\partial N}\int_0^{2\pi} d\varphi\, e^{iN\varphi}|\varphi\rangle = \frac{1}{i}\frac{\partial}{\partial N}|N\rangle$$

hence $\boxed{\hat{\varphi}|N\rangle = \frac{i}{i}\frac{\partial}{\partial N}|N\rangle}$

Explicit derivation of Wavefunction

$$\boxed{\begin{aligned} \psi_\varphi(\varphi) &= \langle\varphi|\psi\rangle \\ \psi_N(N) &= \langle N|\psi\rangle \end{aligned}}$$

Phase distribution
charge (Cooper pair distribution

Hence: $-i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle$ | Multiply: $\langle\varphi|$

$i\hbar\partial_t\langle\varphi|\psi\rangle = i\hbar\partial_t\psi(\varphi) = \langle\varphi|\hat{H}|\psi\rangle$ insert projector

$$= \langle\varphi|\hat{H}|\varphi'\rangle \underbrace{\langle\varphi'|\psi\rangle}_{\psi(\varphi')} d\varphi'$$

$$= \int d\varphi' \langle\varphi|4E_C(\hat{N}-N_G)^2|\varphi\rangle - \langle\varphi|E_J\cos\hat{\varphi}|\varphi\rangle \cdot \psi(\varphi')$$

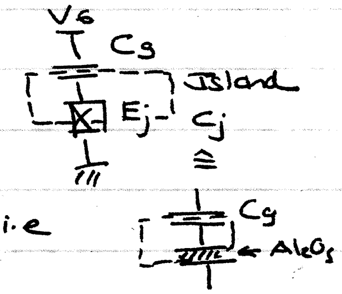
$$= \int d\varphi \left[4E_C(\underbrace{\langle\varphi|\hat{N}|\varphi\rangle}_{\frac{i}{i}\frac{\partial}{\partial\varphi}\ \text{(operator)}} - N_G)^2 - E_J\cos(\underbrace{\langle\varphi|\hat{\varphi}|\varphi\rangle}_{\varphi\ \text{(scalar variable)}})\right]\psi_\varphi(\varphi')$$

$$= \left[4E_C\left(\frac{i}{i}\frac{\partial}{\partial\varphi} - N_G\right)^2 - E_J\cos\varphi\right]\cdot\psi_\varphi(\varphi) = E_n\psi_\varphi(\varphi)$$

$$\Leftrightarrow \boxed{\left[4E_C\left(\frac{1}{i}\frac{\partial}{\partial\varphi} - N_G\right) - E_J\cos\varphi\right]\psi_\varphi(\varphi) = E_n\psi_\varphi(\varphi)}$$ MATTIEU EQUATION

Quantum Mechanics of Phase $\hat{\varphi}$ and Cooper pair Operator $\hat{N}$

(A. COTTET Thesis)



Charge representation: "Number of excess Cooper pairs" → in the metallic island. ⇒ formed by JJ dielectric and gate. $E_C = \frac{(2e)^2}{2C_\Sigma}$ ∧ $C_\Sigma = C_g + C_J$

Cooper pair Coulomb energy

$\hat{N}|N\rangle = N|\hat{N}\rangle$ key point $N \in \mathbb{Z}$ (!) not $\mathbb{N}_0$

[(typical values $C_g \sim 10aF$; $C_j \sim 1fF$) $A_{junction} \sim 100 \times 100nm$]

$$\hat{H}_{el} = \hat{E}_c(\hat{N} - N_g)^2 \qquad N_g \equiv \frac{VC_g}{2e}$$ (2e: cooper pair charge)

$$\hat{H}_j = -\frac{E_j}{2}\sum_{N\in\mathbb{Z}} (|N\rangle\langle N+1| + |N+1\rangle\langle N|)$$

⇒ $$\boxed{\hat{H}(N_g) = \sum_{N\in\mathbb{Z}} E_c(\hat{N} - N_G)^2 - \frac{E_j}{2}(|N\rangle\langle N+1| + |N+1\rangle\langle N|)}$$ charge Basis

Solution: $\hat{H}(N_g)|k\rangle = E_k|k\rangle$ and N_g is CONTINUOUS VARIABLE.

⇒ $E_c = E_j$ ie $E_j/E_c = 1$ case..